The derivative of the idealised edge (step function)

The Heaviside step function is a discontinuous function whose value is zero for a negative argument and 1 for a positive argument –

$$H(x) = 0 \qquad x < 0$$
$$H(x) = 1 \qquad x \geq 1$$

To within an overall proportionality factor, it is a good mathematical representation of an ideal edge running along the y axis of an image.

There are a number of related ways to calculate the derivative of the Heaviside step function. We will use a limiting form based on a smooth function which approximates the step function increasingly well as a scaling parameter $k \to \infty$

$$H(x) = \lim_{k\to\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(kx)$$

So, $\dfrac{dH}{dx} = \lim_{k\to\infty} \dfrac{d}{dx}\left(\dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}(kx)\right) = \dfrac{1}{\pi}\lim_{k\to\infty}\dfrac{d}{dx}\left(\tan^{-1}(kx)\right)$

We can set $y = \tan^{-1}(kx)$ and thus $kx = \tan y$ to obtain

$$\frac{dH}{dx} = \frac{1}{\pi}\lim_{k\to\infty}\left(\frac{k}{1+k^2x^2}\right)$$

Examining the properties of this function, we see that

$$\frac{dH}{dx} \to \infty \qquad \text{for } x = 0$$
$$\frac{dH}{dx} \to 0 \qquad \text{for } x \neq 0$$
$$\int_{-\infty}^{\infty}\left(\frac{dH}{dx}\right)dx = \int_{-\infty}^{\infty}\left(\frac{k}{1+k^2x^2}\right)dx = 1$$

Thus $\dfrac{dH}{dx} = \delta(x)$, the dirac delta function.