The Wiener-Helstrom filter

The (linear, shift invariant) imaging equation is expressed in the spatial domain as –

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,h(x-x',y-y')\,dx'dy' + n(x,y) \qquad (1)$$

where $g(x,y)$ is the recorded image, $f(x,y)$ is the corresponding input distribution, $h(x-x',y-y')$ is the point spread function and $n(x,y)$ is the additive noise.

Use of the convolution theorem gives the frequency domain equivalent directly as –

$$G(k_x,k_y) = H(k_x,k_y)\,F(k_x,k_y) + N(k_x,k_y) \qquad (2)$$

The basic image restoration problem is to form some estimate of the input distribution $\hat{f}(x,y)$ that approximates the actual distribution $f(x,y)$ as closely as possible. An intuitive and well-defined measure is to choose $\hat{f}(x,y)$ so as to minimise the overall mean square error defined as –

$$Q = \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\hat{f}(x,y) - f(x,y)\right]^2 dxdy \right\rangle \qquad (3)$$

where the angle brackets denote averaging over any stochastic variation in the estimate $\hat{f}(x,y)$.

The Wiener-Helstrom filter starts by transforming equation (3) into its frequency domain equivalent. Parseval's theorem states that –

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[f(x,y)\right]^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[F(k_x,k_y)\right]^2 dk_x dk_y \qquad (4)$$

Applying this to the two terms in eq. (3), we therefore obtain –

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle \left[\hat{F}(k_x,k_y) - F(k_x,k_y)\right]^2 \right\rangle dk_x dk_y \qquad (5)$$

The Wiener-Helstrom filter is *linear* and we therefore posit a filter $Y(k_x,k_y)$ such that our estimate is formed by multiplying the output spectrum by the filter -

$$\hat{F}(k_x,k_y) = Y(k_x,k_y)\,G(k_x,k_y) \qquad (6)$$

Substituting equations (2) and (6) into equation (5) (and dropping the explicit dependence of each term on k_x and k_y for brevity), we obtain –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y \left\langle (HF+N)^* Y^* Y(HF+N) - Y(HF+N)F^* - F(HF+N)^* Y^* + F^* F \right\rangle \tag{7}$$

We note at this juncture that the input distribution and the noise are the unknown quantities and thus treated as stochastic. Making use of the relation $(AB)^* = B^* A^*$ equation (7) may be expressed as –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y \left\langle (F^*H^*+N^*)Y^*Y(HF+N) - Y(HF+N)F^* - F(F^*H^*+N^*)Y^* + F^*F \right\rangle$$

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y \begin{Bmatrix} \langle F^*H^*Y^*YHF\rangle + \langle F^*H^*Y^*YN\rangle + \langle N^*Y^*YHF\rangle + \langle N^*Y^*YN\rangle - \\ \langle YHFF^*\rangle - Y\langle NF^*\rangle - \langle FF^*H^*Y^*\rangle - \langle FN^*\rangle Y^* + F^*F \end{Bmatrix} \tag{8}$$

We can factor out the non-stochastic terms from the averaging brackets to write this as –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y \begin{Bmatrix} Y^*Y|H|^2\langle F^*F\rangle + Y^*YH^*\langle F^*N\rangle + Y^*YH\langle N^*F\rangle + Y^*Y\langle N^*N\rangle - \\ YH\langle FF^*\rangle - Y\langle NF^*\rangle - H^*Y^*\langle FF^*\rangle - Y^*\langle FN^*\rangle + F^*F \end{Bmatrix} \tag{9}$$

It is a reasonable assumption to assume that there is no statistical correlation between the unknown input distribution represented by *F* and the noise process represented by *N*. Certainly, there is no physical basis for this in real imaging systems. Accordingly, we may set $\langle NF^*\rangle = \langle FN^*\rangle = 0$. Equation (8) then reduces to –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y \left(Y^*Y|H|^2 W_F + Y^*YW_N - YHW_F - H^*Y^*W_F + W_F \right) \tag{10}$$

Where the noise power spectrum $W_N(k_x,k_y) = |N(k_x,k_y)|^2$ and the input power spectrum $W_F(k_x,k_y) = |F(k_x,k_y)|^2$

There are two possible (and in fact equivalent) routes to minimising *Q* in equation 9 for the unknown filter *Y*. For the reader familiar with methods of variational calculus, we note that the integrand in eq (9) is a **functional,** in which we seek that function Y which will minimise the integral. In general this functional may have explicit dependencies on Y, derivatives of Y and the spatial frequency variables k_x, k_y, we thus write -

$$L\ \ Y\ \ k_x, k_y\ \ , Y'_{k_x}, Y'_{k_y}; k_x, k_y\ \ = Y^*Y\left|H\right|^2 W_F + Y^*YW_N - YHW_F - H^*Y^*W_F + W_F$$
$$\text{where } Y'_{k_x} = \frac{\partial Y}{\partial k_x} \text{ and } Y'_{k_y}\ \frac{\partial Y}{\partial k_y} \tag{11}$$

We can then use the standard Euler-Lagrange solution for a single function of 2 variables, namely –

$$\frac{\partial L}{\partial Y^*} - \frac{d}{dk_x}\left(\frac{\partial L}{\partial Y'_{k_x}}\right) - \frac{d}{dk_y}\left(\frac{\partial L}{\partial Y'}\right) = 0 \tag{12}$$

Since the functional L has no explicit dependence on Y'_{k_x} and Y'_{k_y}, this reduces simply to –

$$\frac{\partial L}{\partial Y} = 0 \tag{13}$$

Thus,

$$L = Y^*Y\left|H\right|^2 W_F + Y^*YW_N - YHW_F - H^*Y^*W_F + W_F$$
$$\frac{\partial L}{\partial Y} = Y^*\left|H\right|^2 W_F + Y^*W_N - HW_F = 0 \tag{14}$$

Yielding the solution

$$Y = \frac{H^*W_F}{\left|H\right|^2 W_F + W_N} \tag{15}$$

A second more direct approach to the solution in this particular case is to consider small perturbation/variations in the function Y, $Y \rightarrow Y + \delta Y$ and demand that the corresponding variation in the functional as $L \rightarrow L + \delta L$ is stationary (i.e. zero). Starting from equation (11), we thus have -

$$L = Y^*Y\left|H\right|^2 W_F + Y^*YW_N - YHW_F - H^*Y^*W_F + W_F$$
$$\delta L = \delta Y^*\left[Y\left|H\right|^2 W_F + YW_N - H^*W_F\right] + \delta Y\left[Y^*\left|H\right|^2 W_F + Y^*W_N - HW_F\right] = 0 \tag{16}$$

which yields an identical result directly –

$$Y = \frac{H^*W_F}{\left|H\right|^2 W_F + W_N} \tag{17}$$

Substitution of eq. (17) back into eq. (10) shows that the mean-square error obtained is –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y\left\{\frac{|H|^2W_F^2}{|H|^2W_F+W_N}-\frac{|H|^2W_F^2}{|H|^2W_F+W_N}-\frac{|H|^2W_F^2}{|H|^2W_F+W_N}+W_F\frac{|H|^2W_F+W_N}{|H|^2W_F+W_N}\right\}$$

which simplifies to –

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dk_x dk_y\left\{\frac{W_F W_N}{|H|^2W_F+W_N}\right\} \qquad (18)$$