The real Fourier series for a 1D periodic, spatial function of period λ takes the form –

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{2\pi nx}{\lambda} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{\lambda} \tag{1}$$

CALCULATION OF THE $\{a_n\}$

To estimate the coefficients $\{a_n\}$, we multiply both sides of this equation by $\cos\frac{2\pi mx}{\lambda}$ where *m* is integer and integrate over one full period of the function from $\frac{-\lambda}{2}$ to $\frac{\lambda}{2}$ -

$$\int_{-\lambda/2}^{\lambda/2} f(x)\cos\frac{2\pi mx}{\lambda}\,dx = \int_{=\lambda/2}^{\lambda/2}\left[\sum_{n=0}^{\infty} a_n \cos\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda}\right]dx \tag{2}$$

To evaluate the RHS of this equation, we split it into two parts which we will examine separately.

$$\int_{-\lambda/2}^{\lambda/2} f(x)\cos\frac{2\pi mx}{\lambda}\,dx = I_1 + I_2 \tag{3}$$

where

$$I_1 = \sum_{n=0}^{\infty} a_n \left(\int_{=\lambda/2}^{\lambda/2} \cos\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda}\,dx \right) \tag{4}$$

$$I_2 = \sum_{n=1}^{\infty} b_n \left(\int_{=\lambda/2}^{\lambda/2} \sin\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda}\,dx \right) \tag{5}$$

To evaluate the integrals I_1 and I_2, first note the trigonometric identities –

$$\begin{aligned} \cos(A \pm B) &= \cos A\cos B \mp \sin A \sin B \\ \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \end{aligned} \tag{6}$$

These allow us to write –

$$\cos\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda} = \frac{1}{2}\left[\cos\frac{2\pi(n+m)x}{\lambda} + \cos\frac{2\pi(n-m)x}{\lambda}\right] \tag{7}$$

$$\sin\frac{2\pi nx}{\lambda}\cos\frac{2\pi mx}{\lambda} = \frac{1}{2}\left[\sin\frac{2\pi(n+m)x}{\lambda} + \sin\frac{2\pi(n-m)x}{\lambda}\right] \tag{8}$$

Thus

$$I_1 = \sum_{n=0}^{\infty} \frac{a_n}{2} \left(\int_{=\lambda/2}^{\lambda/2} \left[\cos \frac{2\pi(n+m)x}{\lambda} + \cos \frac{2\pi(n-m)x}{\lambda} \right] dx \right) \tag{9}$$

$$I_2 = \sum_{n=1}^{\infty} \frac{b_n}{2} \left(\int_{=\lambda/2}^{\lambda/2} \left[\sin \frac{2\pi(n+m)x}{\lambda} + \sin \frac{2\pi(n-m)x}{\lambda} \right] dx \right) \tag{10}$$

Noting that the integral of sin and cos over any interval that is an integer multiple of 2π is **zero**, we therefore have –

For $m \neq n \quad I_1 + I_2 = 0$

Thus it is only when *m=n* that the sum has a non-zero term. There are two cases to consider -

For $m = n = 0; \qquad I_1 + I_2 = \frac{a_0}{2}(\lambda + \lambda) + \frac{b_0}{2}(0+0)$

Thus

$$a_0 = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x)\, dx \tag{11}$$

For $m = n \neq 0; \qquad I_1 + I_2 = \frac{a_m}{2}(\lambda + 0) + \frac{b_m}{2}(0+0)$

Thus

$$a_m = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x) \cos \frac{2\pi m x}{\lambda}\, dx \tag{12}$$

CALCULATION OF THE $\{b_n\}$

To calculate the values of the coefficients $\{b_n\}$, we proceed in a similar fashion but this time we multiply both sides of the Fourier expansion (eq. 1) by $\sin \frac{2\pi m x}{\lambda}$ and integrate from $\frac{-\lambda}{2}$ to $\frac{\lambda}{2}$. Thus, we have -

$$\int_{-\lambda/2}^{\lambda/2} f(x) \sin \frac{2\pi m x}{\lambda}\, dx = \int_{=\lambda/2}^{\lambda/2} \left[\sum_{n=0}^{\infty} a_n \cos \frac{2\pi n x}{\lambda} \sin \frac{2\pi m x}{\lambda} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n x}{\lambda} \sin \frac{2\pi m x}{\lambda} \right] dx \tag{13}$$

We similarly make use of the relation $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$ to write

$$\sin \frac{2\pi n x}{\lambda} \cos \frac{2\pi m x}{\lambda} = \frac{1}{2} \left[\sin \frac{2\pi(n+m)x}{\lambda} + \sin \frac{2\pi(n-m)x}{\lambda} \right] \tag{14}$$

and we use the relation $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ to write

$$\sin\frac{2\pi mx}{\lambda}\sin\frac{2\pi nx}{\lambda}=\frac{1}{2}\left(\cos\frac{2\pi(m-n)x}{\lambda}-\cos\frac{2\pi(m+n)x}{\lambda}\right) \qquad (15)$$

Thus

$$\begin{aligned}\int_{-\lambda/2}^{\lambda/2} f(x)\sin\frac{2\pi mx}{\lambda}\,dx &= \int_{-\lambda/2}^{\lambda/2}\sum_{n=0}^{\infty}a_n\,\frac{1}{2}\left(\sin\frac{2\pi(n+m)x}{\lambda}+\sin\frac{2\pi(n-m)x}{\lambda}\right)dx \\ &+\int_{-\lambda/2}^{\lambda/2}dx\sum_{n=1}^{\infty}b_n\,\frac{1}{2}\left(\cos\frac{2\pi(m-n)x}{\lambda}-\cos\frac{2\pi(m+n)x}{\lambda}\right)dx\end{aligned} \qquad (16)$$

The integration of sin and cos over any interval that is a multiple of 2π is **zero**[1] and we further note that the first term is zero for any value of *n* and *m*.

For the 2nd term, we have a non-zero value only if $m=n\neq 0$. This yields

$$\int_{-\lambda/2}^{\lambda/2} f(x)\sin\frac{2\pi mx}{\lambda}\,dx=\frac{b_m}{2}\int_{-\lambda/2}^{\lambda/2}\cos 0\,dx$$

$$b_m=\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\sin\frac{2\pi mx}{\lambda}\,dx \qquad (17)$$

[1] The total area under the curve is zero, ok ?