Chapter 5: The Frequency Domain

Questions and Exercises

1. In the complex Fourier expansion of a 1-D function

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n \exp(ik_n x) \qquad where\ k_n = \frac{2\pi n}{\lambda}$$

The expansion coefficients are given by the expression

$$c_n = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\exp(-ik_n x)\,dx$$

Use the corresponding expressions for the expansion coefficients a_n and b_n in the real Fourier series to show that –

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \qquad n > 0 \\ b_n &= i(c_n - c_{-n}) \qquad n > 0 \end{aligned}$$

2. Calculate the Fourier series representation for the half-wave rectified sinusoid –

$$\begin{aligned} f(x) &= \sin\left(\frac{2\pi x}{\lambda}\right) \qquad \text{for } n\lambda \le x \le \left(n+\frac{1}{2}\right)\lambda\ ;\ n = 0,1,2\cdots \\ f(x) &= 0 \qquad otherwise \end{aligned}$$

Modify the Fourier series example in the book to create a Matlab program which calculates the first 10 terms in the series and plots the series approximation to the function.

3. Read in the image *basic_shapes.png*. Using the Matlab function ***ginput*** read in about 12 coordinate pairs, approximately equidistantly spaced, from around the boundary of one of the shapes. Calculate a radial Fourier series approximation to the shape using n=5 terms and plot the approximate shape next to the sampled coordinates. Repeat this exercise for a different shape within the image.

4. The 1-D Gaussian or (normal) density function is of the form –

$$f(x) = \frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\alpha^2 x^2\right)$$

Write down the definition of its Fourier transform. By completing the square in the argument of the exponential function show that the Gaussian function Fourier transforms to a Gaussian function of the form –

$$F(k_x) = \exp\left(-\frac{1}{2}\frac{k_x^2}{(2\alpha)^2}\right)$$

How does this equation tell you that a ‘thin’ Gaussian Fourier transforms to a ‘fat’ Gaussian and vice-versa ?

3. The 1-D rectangle function is defined as –

$$\begin{aligned} \mathrm{rect}\left(\frac{x}{a}\right) &= 1 \qquad |x| < \frac{a}{2} \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

   Calculate the Fourier transform of this function analytically.

   Use Matlab to represent a rectangle function $\mathrm{rect}(x/a)$ with *a=1* at 256 sample points over the range $-2.5 < x < 2.5$. Use the Matlab function *fft* to calculate its digital Fourier transform. Plot the rectangle function and its Fourier transform side by side. Repeat this for different values of *a*. What do you observe ?

4. Use Matlab to represent a 2-D rectangular aperture defined by $\mathrm{rect}(x/a)\,\mathrm{rect}(y/b)$ with $a = b = 1$ over a 512^2 array over the range $-2.5 < x < 2.5; -2.5 < y < 2.5$. Use the Matlab function *fft2* to calculate its digital Fourier transform. Display the aperture and the magnitude of its Fourier transform side by side. Experiment for different values of *a* and *b*. What do you observe ?

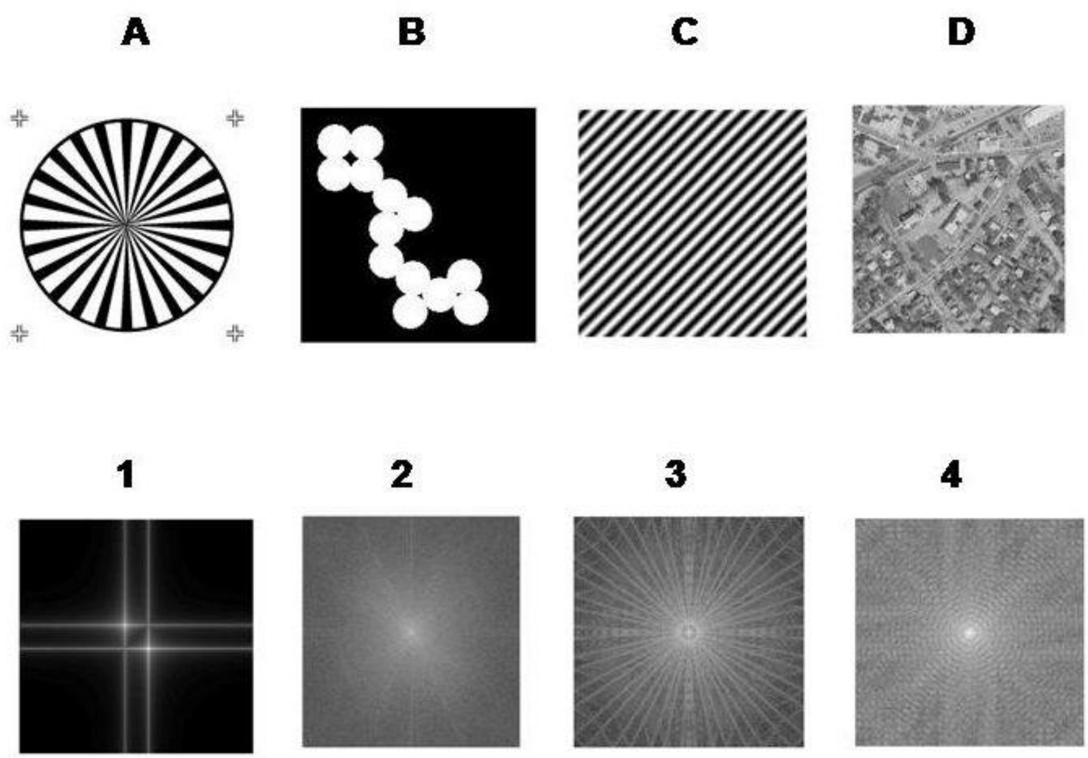
5. Circular apertures which have a transmission value of 1 inside the aperture and value 0 outside are common in computer simulation and modelling of basic diffraction phenomena and optical systems. Diffraction and the optics of the scalar electromagnetic field can in fact be treated very effectively through Fourier transform relationships (see for example ‘Introduction to Fourier Optics’, J.W. Goodman).

   Use Matlab to define a 2-D circular aperture of radius 20 pixels whose centre is at the centre of a 512^2 array. (You may wish to make use of the function ***cart2pol*** and ***pol2cart*** for transforming from Cartesian to polar coordinates and vice-versa)

Calculate the Fourier transform of the aperture using the Matlab function ***fft2** and* display the magnitude. Then apply the Matlab function fftshift to the output of ***fft2*** and display the magnitude again. What do you observe ?

Finally, display the square magnitude of the shifted Fourier transform as a 2-D image. The result is something called a Bessel function, well known to optical scientists and astronomers accustomed to observing

6. In figure 1, 4 images labelled A,B,C and D are displayed. The images labelled 1,2,3 and 4 show the Fourier spectra of these images but the order in which they are displayed is random.
Match each image with its corresponding Fourier spectrum, briefly stating your reasons as appropriate.
Note that the Fourier spectra are displayed as $\log(1+|\mathbf{F}|)$ where $\mathbf{F}$ is the Fourier transform of the image.



7.
If we calculate the Fourier decomposition of a 2-D image $I(x,y)$ through the Fourier transform, what basic meaning can we attach to the transform data ? What does it represent ?

8.

Three, discrete linear convolution filters A and B and C are given as –

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

The image I1 below has been independently processed three times to produce the results labelled I2, I3 and I4 shown alongside. Load the image trui.png into Matlab and process the image using each of these filters to match each of them to the processed images.

I1

I2

I3

I4